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INITIAL IDEALS AND NORMALIZED VOLUMES OF CERTAIN CONVEX POLYTOPES RELATED WITH ROOT SYSTEMS

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ABSTRACT. The present paper is a brief draft of the paper [9]. Let $\Phi \subset \mathbb{Z}^n$ denote one of the classical irreducible root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$, and write $\Phi^{(+)}$ for the configuration consisting of all positive roots of Φ together with the origin of $\mathbb{R}^n$. In [4], by constructing an explicit unimodular triangulation, Gelfand, Graev and Postnikov showed that the normalized volume of the convex hull of $\mathbf{A}_{n-1}^{(+)}$ is equal to the Catalan number. On the other hand, Fong [3] computed the normalized volume of the convex hull of each of the configurations $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$. Moreover, the normalized volume of the convex hull of the subconfiguration of $\mathbf{A}_{n-1}^{(+)}$ arising from a complete bipartite graph was computed by [7] and [3]. The purpose of the present paper is, via the theory of Gröbner bases of toric ideals and triangulations, to compute the normalized volume of the convex hull of each of the subconfigurations of $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$ arising from a complete bipartite graph.

INTRODUCTION

A *configuration* in $\mathbb{R}^n$ is a finite set $\mathcal{A} \subset \mathbb{Z}^n$. Let $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ denote the Laurent polynomial ring $K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ over a field K. We associate a configuration $\mathcal{A} \subset \mathbb{Z}^n$ with the homogeneous semigroup ring $\mathcal{R}_K[\mathcal{A}] = K[\{\mathbf{t}^{\mathbf{a}}s \,;\, \mathbf{a} \in \mathcal{A}\}]$, the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by all monomials $\mathbf{t}^{\mathbf{a}}s$ with $\mathbf{a} \in \mathcal{A}$, where $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_n^{a_n}$ if $\mathbf{a} = (a_1, \ldots, a_n)$. Let $K[\mathcal{A}] = K[\{x_{\mathbf{a}} \,;\, \mathbf{a} \in \mathcal{A}\}]$ denote the polynomial ring over K in the variables $x_{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{A}$, where each $\deg x_{\mathbf{a}} = 1$. The *toric ideal* $I_{\mathcal{A}}$ of $\mathcal{A}$ is the kernel of the surjective homomorphism $\pi : K[\mathcal{A}] \to \mathcal{R}_K[\mathcal{A}]$ defined by setting $\pi(x_{\mathbf{a}}) = \mathbf{t}^{\mathbf{a}}s$ for all $\mathbf{a} \in \mathcal{A}$.

Let $\mathrm{conv}(\mathcal{A})$ denote the convex hull of of $\mathcal{A}$. An *abstract simplex* of $\mathcal{A}$ is a subset $\mathcal{A}' \subset \mathcal{A}$ such that $\mathrm{conv}(\mathcal{A}') \subset \mathbb{R}^n$ is a simplex of dimension $|\mathcal{A}'| - 1$, where $|\mathcal{A}'|$ is the cardinality of the finite set $\mathcal{A}'$. In other words, $\mathcal{A}' \subset \mathcal{A}$ is an abstract simplex of $\mathcal{A}$ if $\{\mathbf{t}^{\mathbf{a}}s \,;\, \mathbf{a} \in \mathcal{A}'\}$ is algebraically independent over K. Let δ denote the dimension of the convex polytope $\mathrm{conv}(\mathcal{A})$. Thus $\delta + 1$ is the maximal cardinality of abstract simplices of $\mathcal{A}$. If, in general, $\mathcal{B}$ is a subset of $\mathcal{A}$, then we write $\mathbb{Z}\mathcal{B}$ for the additive group $\sum_{\mathbf{a} \in \mathcal{B}} \mathbb{Z}\mathbf{a}$ ($\subset \mathbb{Z}^n$) with $\mathcal{B}$ of its system of generators. The *normalized volume* $\mathrm{vol}_{\mathcal{A}}(\mathcal{A}')$ of an abstract simplex $\mathcal{A}'$ of $\mathcal{A}$ with $|\mathcal{A}'| = \delta + 1$ is the index $[\mathbb{Z}\mathcal{A} : \mathbb{Z}\mathcal{A}']$ of $\mathbb{Z}\mathcal{A}'$ in $\mathbb{Z}\mathcal{A}$. A *triangulation* of $\mathcal{A}$ is a collection Δ of abstract simplices of $\mathcal{A}$ satisfying the following conditions:

(i) if $\mathcal{A}' \in \Delta$, then all subsets of $\mathcal{A}'$ belong to Δ;
(ii) if $\mathcal{A}'$ and $\mathcal{A}''$ belong to Δ, then $\mathrm{conv}(\mathcal{A}' \cap \mathcal{A}'') = \mathrm{conv}(\mathcal{A}') \cap \mathrm{conv}(\mathcal{A}'')$;
(iii) $\mathrm{conv}(\mathcal{A}) = \bigcup_{\mathcal{A}' \in \Delta} \mathrm{conv}(\mathcal{A}')$.

A triangulation Δ of $\mathcal{A}$ is called *unimodular* if the normalized volume $\mathrm{vol}_{\mathcal{A}}(\mathcal{A}')$ of each $\mathcal{A}' \in \Delta$ with $|\mathcal{A}'| = \delta + 1$ is equal to 1. The normalized volume of the configuration $\mathcal{A}$ itself may be defined to be the positive integer

$$\sum_{\mathcal{A}' \in \Delta;\, |\mathcal{A}'| = \delta+1} \mathrm{vol}_{\mathcal{A}}(\mathcal{A}'),$$

which is independent of a choice of triangulations Δ of $\mathcal{A}$. See [10, p. 36].

We consider the configurations arising from the classical irreducible root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$ ([5, pp. 64 – 65]). Let $\Phi \subset \mathbb{Z}^n$ denote one of these root systems and write $\Phi^{(+)}$ for the configuration consisting of all positive roots of Φ together with the origin of $\mathbb{R}^n$. More explicitly,

$$\begin{aligned}
\mathbf{A}_{n-1}^{(+)} &= \{\mathbf{0}\} \cup \{\mathbf{e}_i - \mathbf{e}_j \,;\, 1 \le i < j \le n\},\\
\mathbf{B}_{n}^{(+)} &= \mathbf{A}_{n-1}^{(+)} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \cup \{\mathbf{e}_i + \mathbf{e}_j \,;\, 1 \le i < j \le n\},\\
\mathbf{C}_{n}^{(+)} &= \mathbf{A}_{n-1}^{(+)} \cup \{2\mathbf{e}_1, \ldots, 2\mathbf{e}_n\} \cup \{\mathbf{e}_i + \mathbf{e}_j \,;\, 1 \le i < j \le n\},\\
\mathbf{D}_{n}^{(+)} &= \mathbf{A}_{n-1}^{(+)} \cup \{\mathbf{e}_i + \mathbf{e}_j \,;\, 1 \le i < j \le n\}.
\end{aligned}$$

Here $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbb{R}^n$ and $\mathbf{0}$ is the origin of $\mathbb{R}^n$.

Let $[n] = \{1, \ldots, n\}$ denote the vertex set and Σ a finite connected graph on $[n]$ having no loop and no multiple edge. Let $E(\Sigma)$ denote the set of edges of Σ. For each $e = \{i, j\} \in E(\Sigma)$ with $i < j$, let $\rho(e) = \mathbf{e}_i + \mathbf{e}_j \in \mathbb{Z}^n$ and $\delta(e) = \mathbf{e}_i - \mathbf{e}_j \in \mathbb{Z}^n$. The research object in the present paper is the configurations

$$\begin{aligned}
A_{n-1}(\Sigma) &= \{\mathbf{0}\} \cup \{\delta(e) \,;\, e \in E(\Sigma)\},\\
B_n(\Sigma) &= \{\mathbf{0}\} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \cup \{\rho(e) \,;\, e \in E(\Sigma)\} \cup \{\delta(e) \,;\, e \in E(\Sigma)\},\\
C_n(\Sigma) &= \{\mathbf{0}\} \cup \{2\mathbf{e}_1, \ldots, 2\mathbf{e}_n\} \cup \{\rho(e) \,;\, e \in E(\Sigma)\} \cup \{\delta(e) \,;\, e \in E(\Sigma)\},\\
D_n(\Sigma) &= \{\mathbf{0}\} \cup \{\rho(e) \,;\, e \in E(\Sigma)\} \cup \{\delta(e) \,;\, e \in E(\Sigma)\}.
\end{aligned}$$

When Σ is the complete graph on $[n]$, these configurations coincide with $\mathbf{A}_{n-1}^{(+)}$, $\mathbf{B}_{n}^{(+)}$, $\mathbf{C}_{n}^{(+)}$ and $\mathbf{D}_{n}^{(+)}$, respectively. (The complete graph on $[n]$ is the finite graph on $[n]$ such that the set of its edges is equal to $\{\{i, j\} \,;\, 1 \le i < j \le n\}$.)

In [4], by constructing an explicit unimodular triangulation, Gelfand, Graev and Postnikov showed that the normalized volume of the convex hull of $\mathbf{A}_{n-1}^{(+)}$ is equal to the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$. On the other hand, Fong [3] computed the normalized volume of the convex hull of the configurations $\mathbf{B}_{n}^{(+)}$, $\mathbf{C}_{n}^{(+)}$ and $\mathbf{D}_{n}^{(+)}$. Moreover, if Σ is the complete bipartite graph on $[n] = [n_1 + n_2]$ with $E(\Sigma) = \{\{i, j\}; 1 \le i \le n_1, n_1 + 1 \le j \le n_1 + n_2\}$, it is known [3, p. 74] and [7, Corollary 2.7] that the normalized volume of the configuration $A_{n-1}(\Sigma) \subset \mathbb{Z}^{n_1+n_2}$ is $\binom{n_1+n_2-2}{n_1-1}$.

The main purpose of the present paper is ,via the theory of initial ideals and triangulations, to compute the normalized volume of the convex hull of the configuration $B_n(\Sigma)$, $C_n(\Sigma)$ and $D_n(\Sigma)$ when Σ is a complete bipartite graph on $[n]$.

We here review basic facts on the theory of initial ideals and triangulations. Work with the same notation $\mathcal{A} \subset \mathbb{Z}^n$, $K[\mathbf{t}, \mathbf{t}^{-1}, s]$, $\mathcal{R}_K[\mathcal{A}]$, $K[\mathcal{A}]$, $\pi : K[\mathcal{A}] \to \mathcal{R}_K[\mathcal{A}]$ and $I_{\mathcal{A}}$ as before. Let $\mathcal{M}(K[\mathcal{A}])$ denote the set of monomials of $K[\mathcal{A}]$. In particular,

$1 \in \mathcal{M}(K[\mathcal{A}])$. A *monomial order* $<$ on $K[\mathcal{A}]$ is a total order on $\mathcal{M}(K[\mathcal{A}])$ such that (i) $1 < u$ for all $1 \neq u \in \mathcal{M}(K[\mathcal{A}])$ and (ii) for $u, v, w \in \mathcal{M}(K[\mathcal{A}])$, if $u < v$ then $uw < vw$. A *lexicographic order* (resp. *reverse lexicographic order*) on $K[\mathcal{A}]$ induced by the ordering of the variables $x_{\mathbf{a}_1} < x_{\mathbf{a}_2} < \cdots$ of $K[\mathcal{A}]$ is the monomial order $<_{lex}$ (resp. $<_{rev}$) on $K[\mathcal{A}]$ such that, for $u, v \in \mathcal{M}(K[\mathcal{A}])$ with $u = x_{\mathbf{a}_{i_1}} x_{\mathbf{a}_{i_2}} \cdots x_{\mathbf{a}_{i_p}}$ and $v = x_{\mathbf{a}_{j_1}} x_{\mathbf{a}_{j_2}} \cdots x_{\mathbf{a}_{j_q}}$, where $i_1 \leq i_2 \leq \cdots \leq i_p$ and $j_1 \leq j_2 \leq \cdots \leq j_q$, one has $u <_{lex} v$ (resp. $u <_{rev} v$) if $i_k < j_k, i_{k+1} = j_{k+1}, \ldots, i_p = j_p$ (resp. either (i) $p < q$ or (ii) $p = q$ and $i_1 = j_1, i_2 = j_2, \ldots, i_k > j_k$) for some $1 \leq k \leq p$.

Fix a monomial order $<$ on $K[\mathcal{A}]$. The *initial monomial* $in_<(f)$ of $0 \neq f \in I_{\mathcal{A}}$ with respect to $<$ is the biggest monomial appearing in f with respect to $<$. The *initial ideal* of $I_{\mathcal{A}}$ with respect to $<$ is the ideal $in_<(I_{\mathcal{A}})$ of $K[\mathcal{A}]$ generated by all initial monomials $in_<(f)$ with $0 \neq f \in I_{\mathcal{A}}$.

One of the most fundamental facts on the initial ideal $in_<(I_{\mathcal{A}})$ is that

$$\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathcal{A}]), u \notin in_<(I_{\mathcal{A}})\}$$

is a K-basis of $\mathcal{R}_K[\mathcal{A}]$.

If, in general, $\mathcal{G}$ is a finite subset of $I_{\mathcal{A}}$, then we write $in_<(\mathcal{G})$ for the ideal $(in_<(g)\,;\, g \in \mathcal{G})$ of $K[\mathcal{A}]$. A finite subset $\mathcal{G}$ of $I_{\mathcal{A}}$ is said to be a *Gröbner basis* of $I_{\mathcal{A}}$ with respect to $<$ if $in_<(\mathcal{G}) = in_<(I_{\mathcal{A}})$.

Dickson's Lemma [2, p. 69], which says that any nonempty subset of $\mathcal{M}(K[\mathcal{A}])$ (in particular, $in_<(I_{\mathcal{A}}) \cap \mathcal{M}(K[\mathcal{A}])$) has only finitely many minimal elements in the partial order by divisibility, guarantees that a Gröbner basis of $I_{\mathcal{A}}$ with respect to $<$ always exists. Moreover, if $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}}$, then $I_{\mathcal{A}}$ is generated by $\mathcal{G}$.

A Gröbner basis $\mathcal{G}$ of $I_{\mathcal{A}}$ with respect to $<$ is called *quadratic* if each $in_<(g)$ with $g \in \mathcal{G}$ is a quadratic monomial.

Even though the following fundamental facts on Gröbner bases are well-known (e.g., [1, Lemma 1.1] and [6, Proposition 1.1]) and, in fact, can be easily proved, these techniques play important roles throughout the present paper.

Lemma 0.1. *A finite subset $\mathcal{G}$ of $I_{\mathcal{A}}$ is a Gröbner basis of $I_{\mathcal{A}}$ with respect to $<$ if and only if $\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathcal{A}]), u \notin in_<(\mathcal{G})\}$ is linearly independent over K; in other words, if and only if $\pi(u) \neq \pi(v)$ for all $u \notin in_<(\mathcal{G})$ and $v \notin in_<(\mathcal{G})$ with $u \neq v$.*

Lemma 0.2. *Let $\mathcal{B}$ be a subconfiguration of $\mathcal{A}$, $K[\mathcal{B}] = K[\{x_{\mathbf{a}}\,;\, \mathbf{a} \in \mathcal{B}\}]$ $(\subset K[\mathcal{A}])$ and $I_{\mathcal{B}}$ $(= I_{\mathcal{A}} \cap K[\mathcal{B}])$ the toric ideal of $\mathcal{B}$. Let $\mathcal{G}$ be a Gröbner basis of $I_{\mathcal{A}}$ with respect to $<$ and suppose that, for each $g \in \mathcal{G}$ with $in_<(g) \in K[\mathcal{B}]$, one has $g \in K[\mathcal{B}]$. Then $\mathcal{G} \cap I_{\mathcal{B}}$ is a Gröbner basis of $I_{\mathcal{B}}$ (with respect to the monomial order on $K[\mathcal{B}]$ obtained by restricting $<$ to $\mathcal{M}(K[\mathcal{B}])$).*

Let $\sqrt{in_<(I_{\mathcal{A}})}$ denote the radical of the initial ideal $in_<(I_{\mathcal{A}})$. Write $\Delta(in_<(I_{\mathcal{A}}))$ for the set of those subsets $\mathcal{A}' \subset \mathcal{A}$ with

$$\prod_{\mathbf{a} \in \mathcal{A}'} x_{\mathbf{a}} \notin \sqrt{in_<(I_{\mathcal{A}})}.$$

It is known [10, Theorem 8.3] that $\Delta(in_<(I_{\mathcal{A}}))$ is a triangulation of $\mathcal{A}$. Such a triangulation is called *regular*. Moreover, [10, Corollary 8.9] says that $\Delta(in_<(I_{\mathcal{A}}))$ is unimodular if and only if $\sqrt{in_<(I_{\mathcal{A}})} = in_<(I_{\mathcal{A}})$. Hence

Lemma 0.3. *If $\sqrt{in_<(I_{\mathcal{A}})} = in_<(I_{\mathcal{A}})$, then the normalized volume of $\mathcal{A}$ coincides with the number of squarefree monomials u of degree $\delta+1$ of $K[\mathcal{A}]$ with $u \notin in_<(I_{\mathcal{A}})$.*

By using the above facts on Gröbner bases and initial ideals together with explicit computations on Gröbner bases discussed in Section 1 and 2, we have the following.

Theorem 0.4. *Let $n \geq 1$ and $m \geq 1$, and let $\Sigma_{n,m}$ denote the complete bipartite graph on $[n+m]$ with $E(\Sigma_{n,m}) = \{\{i,j\}; 1 \leq i \leq n, n+1 \leq j \leq n+m\}$. Then,*

(a) *The normalized volume of $B_{n+m}(\Sigma_{n,m})$ is $\alpha + \beta$, where*

$$\alpha = \sum_{\substack{1\leq i\leq n \\ 1\leq k\leq \ell\leq m}} 2^{m-\ell}\binom{\ell-1}{k-1}\binom{i+m-k-1}{m-k},$$

$$\beta = \sum_{\substack{1\leq i\leq n \\ 1\leq k\leq m}} \binom{m}{k}\binom{i+k-2}{i-1} + 1.$$

(b) *The normalized volume of $D_{n+m}(\Sigma_{n,m})$ is*

$$\sum_{\substack{1\leq i\leq n \\ 1\leq k\leq m}} \binom{m-1}{k-1}\binom{i+k-2}{i-1}\binom{n-i+m-k}{n-i}.$$

(c) *The normalized volume of $C_{n+m}(\Sigma_{n,m})$ is $\alpha + \beta + \gamma$, where*

$$\alpha = \sum_{\substack{1\leq i\leq n \\ 1\leq k\leq m}} \binom{m-1}{k-1}\binom{i+k-2}{i-1}\binom{n-i+m-k}{n-i},$$

$$\beta = \sum_{\substack{1\leq i\leq p\leq n \\ 1\leq k\leq m}} 2^{n-p}\binom{m-1}{k-1}\binom{i+k-2}{i-1}\binom{p-i+m-k}{p-i},$$

$$\gamma = \sum_{\substack{1\leq p\leq q\leq m \\ 1\leq i\leq n \\ 1\leq k\leq m-q+1}} 2^{p-1}\binom{q-1}{p-1}\binom{m-q}{k-1}\binom{i+k-2}{i-1}\binom{n-i+m-p-k}{n-i}.$$

1. Toric ideals $I_{\mathbf{B}_n^{(+)}}$, $I_{\mathbf{C}_n^{(+)}}$ and $I_{\mathbf{D}_n^{(+)}}$

It is known [4] that the toric ideal $I_{\mathbf{A}_{n-1}^{(+)}}$ possesses both a reverse lexicographic quadratic Gröbner basis and a lexicographic quadratic Gröbner basis. In [8] it is proved that each of the toric ideals $I_{\mathbf{B}_n^{(+)}}$, $I_{\mathbf{C}_n^{(+)}}$ and $I_{\mathbf{D}_n^{(+)}}$ possesses a reverse lexicographic quadratic Gröbner basis.

In the present section, we discuss a lexicographic quadratic Gröbner basis of the toric ideal of each of the configurations $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$, in order to study Gröbner bases of the toric ideals of subconfigrations associated with complete bipartite graphs.

Let $\Phi^{(+)} \subset \mathbb{Z}^n$ denote one of the configurations $\mathbf{A}_{n-1}^{(+)}$, $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$. Let $K[\mathbf{A}_{n-1}^{(+)}]$, $K[\mathbf{B}_n^{(+)}]$, $K[\mathbf{C}_n^{(+)}]$ and $K[\mathbf{D}_n^{(+)}]$ denote the polynomial rings

$$\begin{aligned} K[\mathbf{A}_{n-1}^{(+)}] &= K[\{x\} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}], \\ K[\mathbf{B}_n^{(+)}] &= K[\{x\} \cup \{y_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}], \\ K[\mathbf{C}_n^{(+)}] &= K[\{x\} \cup \{e_{i,i}\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}], \\ K[\mathbf{D}_n^{(+)}] &= K[\{x\} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}] \end{aligned}$$

over K. Write $\pi : K[\Phi^{(+)}] \to K[\mathbf{t}, \mathbf{t}^{-1}, s]$ for the homomorphism defined by setting

$$\pi(x) = s, \quad \pi(y_i) = t_i s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

Thus $\pi(K[\Phi^{(+)}]) = \mathcal{R}_K[\Phi^{(+)}]$ and the kernel of π is the toric ideal $I_{\Phi^{(+)}}$. To simplify the notation, we understand $e_{j,i} = e_{i,j}$ in case of $i < j$.

(1.1) *Toric ideal* $I_{\mathbf{A}_{n-1}^{(+)}}$.

Even though the theory of Gröbner bases does not appear in [4], it is essentially proved that the set of binomials

$$\begin{aligned} f_{i,k}f_{j,\ell} - f_{i,\ell}f_{j,k}, &\quad i < j < k < \ell, \\ f_{i,j}f_{j,k} - xf_{i,k}, &\quad i < j < k, \end{aligned}$$

is a Gröbner basis of $I_{\mathbf{A}_{n-1}^{(+)}}$ with respect to the lexicographic order $<_{lex}^{a}$ on $K[\mathbf{A}_{n-1}^{(+)}]$ induced by the ordering of the variables

$$\begin{aligned} x &< f_{1,2} < f_{1,3} < \cdots < f_{1,n} < f_{2,3} < f_{2,4} < \cdots < f_{2,n} \\ &< \cdots < f_{n-2,n-1} < f_{n-2,n} < f_{n-1,n}. \end{aligned}$$

See [4, Theorem 6.6]. However, for the sake of completeness, based on Lemma 0.1 in Introduction, a simple and quick proof of this fact will be given below.

First of all, each binomial $f_{i,k}f_{j,\ell} - f_{i,\ell}f_{j,k}$ (resp. $f_{i,j}f_{j,k} - xf_{i,k}$) belongs to $I_{\mathbf{A}_{n-1}^{(+)}}$ with $f_{i,k}f_{j,\ell}$ (resp. $f_{i,j}f_{j,k}$) of its initial monomial. Let $\mathcal{G}$ denote the set of binomials $f_{i,k}f_{j,\ell} - f_{i,\ell}f_{j,k}$ and $f_{i,j}f_{j,k} - xf_{i,k}$ listed above and $in_{<_{lex}^{a}}(\mathcal{G}) = (in_{<_{lex}^{a}}(g)\,;\, g \in \mathcal{G})$. Let $\mathcal{M}(K[\mathbf{A}_{n-1}^{(+)}])$ denote the set of monomials of $K[\mathbf{A}_{n-1}^{(+)}]$. Lemma 0.1 guarantees that the finite set $\mathcal{G}$ turns out to be a Gröbner basis of $I_{\mathbf{A}_{n-1}^{(+)}}$ with respect to $<_{lex}^{a}$ if

$$\{\pi(u)\,;\, u \in \mathcal{M}(K[\mathbf{A}_{n-1}^{(+)}]), u \notin in_{<_{lex}^{a}}(\mathcal{G})\}$$

is linearly independent over K. Thus our work is to prove that if

$$\begin{aligned} u &= x^{\alpha} f_{i_1,j_1} \cdots f_{i_q,j_q}, \\ u' &= x^{\alpha'} f_{i'_1,j'_1} \cdots f_{i'_{q'},j'_{q'}} \end{aligned}$$

belong to $\mathcal{M}(K[\mathbf{A}_{n-1}^{(+)}])$ with $u \notin in_{<_{lex}^{a}}(\mathcal{G})$ and $u' \notin in_{<_{lex}^{a}}(\mathcal{G})$, where

$$\begin{aligned} f_{i_1,j_1} \leq_{lex}^{a} \cdots \leq_{lex}^{a} f_{i_q,j_q}, \\ f_{i'_1,j'_1} \leq_{lex}^{a} \cdots \leq_{lex}^{a} f_{i'_{q'},j'_{q'}}, \end{aligned}$$

and if $\pi(u) = \pi(u')$, then $\alpha = \alpha'$, $q = q'$, $f_{i_1,j_1} = f_{i'_1,j'_1}, \ldots, f_{i_q,j_q} = f_{i'_q,j'_q}$.

Since $f_{i,j}f_{j,k} \in in_{<^a_{lex}}(\mathcal{G})$ if $i < j < k$, it follows that, for each $1 \leq i \leq n$, both t_i and t_i^{-1} cannot appear in the product

$$\pi(u) = \pi(x)^\alpha \pi(f_{i_1,j_1}) \cdots \pi(f_{i_q,j_q}).$$

Thus $\alpha = \alpha'$ and $q = q'$. Let $\alpha = \alpha' = 0$ and $q = q' > 1$. Since $i_1 \leq i_2 \leq \cdots \leq i_q$ and $i'_1 \leq i'_2 \leq \cdots \leq i'_q$, it follows that $i_q = i'_q$. If, say, $j_q < j'_q$, then there is $1 \leq h < q'$ with $j_q = j'_h$. Since $f_{i'_h,j'_h} \leq^a_{lex} f_{i'_q,j'_q}$, one has $i'_h \leq i'_q$. Thus $i'_h \leq i'_q < j'_h < j'_q$. Since $f_{i,k}f_{j,\ell} \in in_{<^a_{lex}}(\mathcal{G})$ if $i < j < k < \ell$, it follows that $i'_h = i'_q$, i.e., $f_{i'_h,j'_h} = f_{i_q,j_q}$. Then working with induction on q yields $f_{i'_q,j'_q} = f_{i_{q-1},j_{q-1}}$. However, this contradict $f_{i_{q-1},j_{q-1}}$ $(\leq^a_{lex} f_{i_q,j_q} = f_{i'_h,j'_h}) <^a_{lex} f_{i'_q,j'_q}$. Thus $j_q = j'_q$, i.e., $f_{i_q,j_q} = f_{i'_q,j'_q}$, as desired.

Q. E. D.

On the other hand, it is also easy to prove that the set of binomials

$$\begin{aligned} &f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, && i < j < k < \ell, \\ &f_{i,j}f_{j,k} - xf_{i,k}, && i < j < k, \end{aligned}$$

is a Gröbner basis of $I_{\mathbf{A}_{n-1}^{(+)}}$ with respect to the lexicographic order $<^{a'}_{lex}$ on $K[\mathbf{A}_{n-1}^{(+)}]$ induced by the ordering of the variables

$$\begin{aligned} x &< f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3} \\ &< \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n}. \end{aligned}$$

The proof is as follows.

First, each binomial $f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}$ (resp. $f_{i,j}f_{j,k} - xf_{i,k}$) belongs to $I_{\mathbf{A}_{n-1}^{(+)}}$ with $f_{i,\ell}f_{j,k}$ (resp. $f_{i,j}f_{j,k}$) of its initial monomial. Let $\mathcal{G}$ denote the set of binomials $f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}$ and $f_{i,j}f_{j,k} - xf_{i,k}$ listed above and $in_{<^{a'}_{lex}}(\mathcal{G}) = (in_{<^{a'}_{lex}}(g)\,;\, g \in \mathcal{G})$. Then, our work is to prove that if

$$\begin{aligned} u &= x^\alpha f_{i_1,j_1} \cdots f_{i_q,j_q}, \\ u' &= x^{\alpha'} f_{i'_1,j'_1} \cdots f_{i'_{q'},j'_{q'}} \end{aligned}$$

belong to $\mathcal{M}(K[\mathbf{A}_{n-1}^{(+)}])$ with $u \notin in_{<^{a'}_{lex}}(\mathcal{G})$ and $u' \notin in_{<^{a'}_{lex}}(\mathcal{G})$, where

$$\begin{aligned} &f_{i_1,j_1} \leq^{a'}_{lex} \cdots \leq^{a'}_{lex} f_{i_q,j_q}, \\ &f_{i'_1,j'_1} \leq^{a'}_{lex} \cdots \leq^{a'}_{lex} f_{i'_{q'},j'_{q'}}, \end{aligned}$$

and if $\pi(u) = \pi(u')$, then $\alpha = \alpha'$, $q = q'$, $f_{i_1,j_1} = f_{i'_1,j'_1}, \ldots, f_{i_q,j_q} = f_{i'_q,j'_q}$.

Since $f_{i,j}f_{j,k} \in in_{<^{a'}_{lex}}(\mathcal{G})$ if $i < j < k$, by the same argument as in the proof for $<^a_{lex}$, we have $\alpha = \alpha'$ and $q = q'$. Let $\alpha = \alpha' = 0$ and $q = q' > 1$. Since $i_1 \leq i_2 \leq \cdots \leq i_q$ and $i'_1 \leq i'_2 \leq \cdots \leq i'_q$, it follows that $i_q = i'_q$. If, say, $j_q < j'_q$, then there is $1 \leq h < q$ with $j'_q = j_h$. Since $f_{i_h,j_h} \leq^{a'}_{lex} f_{i_q,j_q}$, one has $i_h \leq i_q$. Thus $i_h \leq i_q < j_q < j_h$. Since $f_{i,\ell}f_{j,k} \in in_{<^{a'}_{lex}}(\mathcal{G})$ if $i < j < k < \ell$, it follows that $i_h = i_q$, i.e., $f_{i_h,j_h} = f_{i'_q,j'_q}$. Then working with induction on q yields $f_{i_q,j_q} = f_{i'_{q-1},j'_{q-1}}$.

However, this contradict $f_{i'_{q-1},j'_{q-1}}$ $(\leq^{a'}_{lex} f_{i'_q,j'_q} = f_{i_h,j_h}) <^{a'}_{lex} f_{i_q,j_q}$. Thus $j_q = j'_q$, i.e., $f_{i_q,j_q} = f_{i'_q,j'_q}$, as desired. Q. E. D.

(1.2) *Toric ideal* $I_{\mathbf{B}_n^{(+)}}$.

Let $<^b_{lex}$ denote the lexicographic order on $K[\mathbf{B}_n^{(+)}]$ induced by the ordering of the variables

$$\begin{aligned} x &< y_1 < y_2 < \cdots < y_n \\ &< e_{1,n} < f_{1,n} < e_{1,n-1} < f_{1,n-1} < \cdots < e_{1,2} < f_{1,2} \\ &< e_{2,n} < f_{2,n} < e_{2,n-1} < f_{2,n-1} < \cdots < e_{2,3} < f_{2,3} \\ &< \cdots \\ &< e_{n-2,n} < f_{n-2,n} < e_{n-2,n-1} < f_{n-2,n-1} \\ &< e_{n-1,n} < f_{n-1,n}. \end{aligned}$$

Theorem 1.1. *The set of the binomials*

$$\begin{array}{rl} e_{i,j}e_{k,\ell} - e_{i,k}e_{j,\ell}, & i<j<k<\ell, \\ e_{i,\ell}e_{j,k} - e_{i,k}e_{j,\ell}, & i<j<k<\ell, \\ f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i<j<k<\ell, \\ f_{i,j}f_{j,k} - xf_{i,k}, & i<j<k, \\ e_{i,j}f_{k,\ell} - e_{i,k}f_{j,\ell}, & i<j<k<\ell, \\ f_{i,\ell}e_{j,k} - e_{i,k}f_{j,\ell}, & i<j<k<\ell, \\ e_{i,\ell}f_{j,k} - f_{i,k}e_{j,\ell}, & i<j<k<\ell, \\ f_{i,j}e_{j,k} - y_iy_k, & i<j,\ \ j \neq k, \\ y_ie_{j,k} - e_{i,k}y_j, & i<j<k, \\ e_{i,j}y_k - e_{i,k}y_j, & i<j<k, \\ y_if_{j,k} - f_{i,k}y_j, & i<j<k, \\ f_{i,j}y_j - y_ix, & i<j, \\ xe_{i,j} - y_iy_j, & i<j, \end{array}$$

is a Gröbner basis of $I_{\mathbf{B}_n^{(+)}}$ *with respect to the lexicographic order* $<^b_{lex}$.

(1.3) *Toric ideal* $I_{\mathbf{C}_n^{(+)}}$.

Let $<^c_{lex}$ denote the lexicographic order on $K[\mathbf{C}_n^{(+)}]$ induced by the ordering of the variables

$$\begin{aligned} x &< e_{1,1} < e_{1,n} < f_{1,n} < e_{1,n-1} < f_{1,n-1} < \cdots < e_{1,2} < f_{1,2} \\ &< e_{2,2} < e_{2,n} < f_{2,n} < e_{2,n-1} < f_{2,n-1} < \cdots < e_{2,3} < f_{2,3} \\ &< \cdots \\ &< e_{n-2,n-2} < e_{n-2,n} < f_{n-2,n} < e_{n-2,n-1} < f_{n-2,n-1} \\ &< e_{n-1,n-1} < e_{n-1,n} < f_{n-1,n} < e_{n,n}. \end{aligned}$$

Theorem 1.2. *The set of the binomials*

$$
\begin{array}{rl}
e_{i,j}e_{k,\ell} - e_{i,k}e_{j,\ell}, & i \leq j < k \leq \ell, \\
e_{i,\ell}e_{j,k} - e_{i,k}e_{j,\ell}, & i < j < k < \ell, \\
e_{i,j}e_{j,k} - e_{i,k}e_{j,j}, & i < j < k, \\
f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i < j < k < \ell, \\
f_{i,j}f_{j,k} - xf_{i,k}, & i < j < k, \\
e_{i,j}f_{k,\ell} - e_{i,k}f_{j,\ell}, & i \leq j < k < \ell, \\
f_{i,\ell}e_{j,k} - e_{i,k}f_{j,\ell}, & i < j < k < \ell, \\
e_{i,j}f_{j,k} - f_{i,k}e_{j,j}, & i < j < k, \\
e_{i,\ell}f_{j,k} - f_{i,k}e_{j,\ell}, & i < j < k < \ell, \\
f_{i,j}e_{j,k} - xe_{i,k}, & i < j,
\end{array}
$$

is a Gröbner basis of $I_{\mathbf{C}_n^{(+)}}$ *with respect to the lexicographic order* $<^c_{lex}$.

(1.4) *Toric ideal* $I_{\mathbf{D}_n^{(+)}}$.

Let $<^d_{lex}$ denote the lexicographic order on $K[\mathbf{D}_n^{(+)}]$ induced by the ordering of the variables

$$
\begin{array}{rl}
x & < \; f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < f_{2,n-1} < \cdots < f_{2,3} \\
& < \; \cdots < f_{n-2,n} < f_{n-2,n-1} < f_{n-1,n} \\
& < \; e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{2,n} < e_{2,n-1} < \cdots < e_{2,3} \\
& < \; \cdots < e_{n-2,n} < e_{n-2,n-1} < e_{n-1,n}.
\end{array}
$$

Theorem 1.3. *The set of the binomials*

$$
\begin{array}{rl}
e_{i,j}e_{k,\ell} - e_{i,k}e_{j,\ell}, & i < j < k < \ell, \\
e_{i,\ell}e_{j,k} - e_{i,k}e_{j,\ell}, & i < j < k < \ell, \\
f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i < j < k < \ell, \\
f_{i,j}f_{j,k} - xf_{i,k}, & i < j < k, \\
e_{i,j}f_{k,\ell} - e_{i,k}f_{j,\ell}, & i < j < k < \ell, \\
f_{i,\ell}e_{j,k} - e_{i,k}f_{j,\ell}, & i < j < k < \ell, \\
f_{i,k}e_{j,\ell} - e_{i,\ell}f_{j,k}, & i < j < k < \ell, \\
e_{i,k}f_{j,k} - e_{i,n}f_{j,n}, & i \leq j < k < n, \\
f_{i,k}e_{j,k} - e_{i,n}f_{j,n}, & i < j < k \leq n, \\
f_{i,k}e_{k,j} - e_{i,n}f_{j,n}, & i < k < j < n, \\
xe_{i,j} - e_{i,n}f_{j,n}, & i < j < n, \\
f_{i,j}e_{j,n} - xe_{i,n}, & i < j < n,
\end{array}
$$

s a Gröbner basis of $I_{\mathbf{D}_n^{(+)}}$ *with respect to the lexicographic order* $<^d_{lex}$.

Remark 1.4. For configurations $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$, the Gröbner bases in [8] are also lexicographic with respect to a certain ordering of variables. However, since the elimination technique (Lemma 0.2) required in Section 2 cannot be applied for the Gröbner bases in [8], the lexicographic Gröbner bases in [8] are not quite useful to find suitable quadratic Gröbner bases of subconfigurations of $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$ related with complete bipartite graphs.

2. Subconfigurations arising from finite graphs

Let $[n] = \{1, \ldots, n\}$ denote the vertex set and Σ a finite connected graph on $[n]$ having no loop and no multiple edge. Let $E(\Sigma)$ denote the set of edges of Σ. For each $e = \{i, j\} \in E(\Sigma)$ with $i < j$, let $\rho(e) = \mathbf{e}_i + \mathbf{e}_j \in \mathbb{Z}^n$ and $\delta(e) = \mathbf{e}_i - \mathbf{e}_j \in \mathbb{Z}^n$. It is reasonable to ask if the toric ideals of the configurations

$$\begin{aligned} A_{n-1}(\Sigma) &= \{\mathbf{0}\} \cup \{\delta(e)\,;\, e \in E(\Sigma)\}, \\ B_n(\Sigma) &= \{\mathbf{0}\} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \cup \{\rho(e)\,;\, e \in E(\Sigma)\} \cup \{\delta(e)\,;\, e \in E(\Sigma)\}, \\ C_n(\Sigma) &= \{\mathbf{0}\} \cup \{2\mathbf{e}_1, \ldots, 2\mathbf{e}_n\} \cup \{\rho(e)\,;\, e \in E(\Sigma)\} \cup \{\delta(e)\,;\, e \in E(\Sigma)\}, \\ D_n(\Sigma) &= \{\mathbf{0}\} \cup \{\rho(e)\,;\, e \in E(\Sigma)\} \cup \{\delta(e)\,;\, e \in E(\Sigma)\} \end{aligned}$$

possess quadratic Gröbner bases. When Σ is the complete graph on $[n]$, these configurations coincide with $\mathbf{A}_{n-1}^{(+)}$, $\mathbf{B}_n^{(+)}$, $\mathbf{C}_n^{(+)}$ and $\mathbf{D}_n^{(+)}$, respectively.

In the present section, first of all, we show that the toric ideals $I_{A_{n-1}(\Sigma)}$ possesses a lexicographic quadratic initial ideal as well as a reverse lexicographic quadratic initial ideal if Σ is a connected graph on $[n]$ satisfying the condition

(2.0) *If $1 \le i \le j < k \le \ell \le n$ and if $\{j, k\} \in E(\Sigma)$, then $\{i, \ell\} \in E(\Sigma)$.*

It seems of difficult to find a combinatorial characterization of connected graphs Σ on $[n]$ such that the toric ideal $I_{A_{n-1}(\Sigma)}$ possesses a quadratic initial ideal. Second, it will be proved that if Σ is a connected graph on $[n]$ satisfying the condition (2.0), then the toric ideals $I_{B_n(\Sigma)}$ possesses a lexicographic quadratic initial ideal. Third, it will be proved that if Σ is a complete bipartite graph on $[n]$, then the toric ideals $I_{D_n(\Sigma)}$ possesses a lexicographic quadratic initial ideal. We also give an example of a complete bipartite graph Σ for which the toric ideal $I_{C_n(\Sigma)}$ cannot be generated by quadratic binomials, and construct a cubic Gröbner basis of $I_{C_n(\Sigma)}$.

Let Σ be a connected graph on $[n]$ and $K[A_{n-1}(\Sigma)]$ the polynomial ring

$$K[A_{n-1}(\Sigma)] = K[\{x\} \cup \{f_{i,j}\}_{\{i,j\} \in E(\Sigma)}]$$

over K. Let $\pi : K[A_{n-1}(\Sigma)] \to K[\mathbf{t}, \mathbf{t}^{-1}, s]$ denote the homomorphism defined by setting

$$\pi(x) = s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

Thus $\pi(K[A_{n-1}(\Sigma)]) = \mathcal{R}_K[A_{n-1}(\Sigma)]$ and the kernel of π is the toric ideal $I_{A_{n-1}(\Sigma)}$. Write $<_{lex}^{aa}$ (resp. $<_{rev}^{aa}$) for the lexicographic (resp. reverse lexicographic) order on $K[A_{n-1}(\Sigma)]$ induced by the ordering of the variables satisfying

(i) $x < f_{i,j}$ for all $\{i, j\} \in E(\Sigma)$;
(ii) $f_{i,j} < f_{i',j'}$ if either (a) $i < i'$, or (b) $i = i'$ and $j > j'$.

Theorem 2.1. *Let Σ be a finite connected graph on $[n]$ and suppose that Σ satisfies the condition (2.0). Then the set of binomials*

$$\begin{array}{ll} f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i < j < k < \ell, \\ f_{i,j}f_{j,k} - xf_{i,k}, & i < j < k \end{array}$$

belonging to $K[A_{n-1}(\Sigma)]$ is a Gröbner basis of $I_{A_{n-1}(\Sigma)}$ with respect to $<^{aa}_{lex}$ as well as with respect to $<^{aa}_{rev}$.

Let Σ be a connected graph on $[n]$ and $K[B_n(\Sigma)]$ the polynomial ring

$$K[B_n(\Sigma)] = K[\{x\} \cup \{y_i\}_{1\leq i \leq n} \cup \{e_{i,j}\}_{\{i,j\}\in E(\Sigma)} \cup \{f_{i,j}\}_{\{i,j\}\in E(\Sigma)}]$$

over K. Let $\pi : K[B_n(\Sigma)] \to K[\mathbf{t}, \mathbf{t}^{-1}, s]$ denote the homomorphism defined by setting

$$\pi(x) = s, \quad \pi(y_i) = t_i s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

Thus $\pi(K[B_n(\Sigma)]) = \mathcal{R}_K[B_n(\Sigma)]$ and the kernel of π is the toric ideal $I_{B_n(\Sigma)}$. Write $<^{bb}_{lex}$ for the lexicographic order on $K[B_n(\Sigma)]$ which is obtained by restricting the lexicographic order $<^{b}_{lex}$ introduced in (1.2) to $\mathcal{M}(K[B_n(\Sigma)])$, i.e., for $u, v \in \mathcal{M}(K[B_n(\Sigma)])$ $(\subset \mathcal{M}(K[\mathbf{B}^{(+)}_n]))$ one has $u <^{bb}_{lex} v$ if and only if $u <^{b}_{lex} v$. It then follows from Theorem 1.1 together with Lemma 0.2 that

Theorem 2.2. *Let $\mathcal{G}$ denote the set of binomials in Theorem 1.1. Let Σ be a finite connected graph on $[n]$ and suppose that Σ satisfies the condition (2.0). Then the set of binomials $\mathcal{G} \cap K[B_n(\Sigma)]$ is a Gröbner basis of $I_{B_n(\Sigma)}$ with respect to $<^{bb}_{lex}$.*

Now, let $n \geq 1$ and $m \geq 1$, and let $\Sigma_{n,m}$ denote the complete bipartite graph on $[n+m]$ with

$$E(\Sigma_{n,m}) = \{\{i,j\}\,;\, 1 \leq i \leq n, n+1 \leq j \leq n+m\}$$

and, to simplify the notation,

$$\begin{array}{rcl} B_{n,m} & = & B_{n+m}(\Sigma_{n,m}) \subset \mathbb{Z}^{n+m}, \\ C_{n,m} & = & C_{n+m}(\Sigma_{n,m}) \subset \mathbb{Z}^{n+m}, \\ D_{n,m} & = & D_{n+m}(\Sigma_{n,m}) \subset \mathbb{Z}^{n+m}. \end{array}$$

Let $\Psi_{n,m} \subset \mathbb{Z}^{n+m}$ denote one of $B_{n,m}$, $C_{n,m}$ and $D_{n,m}$. Let $K[B_{n,m}]$, $K[C_{n,m}]$ and $K[D_{n,m}]$ denote the polynomial rings

$$\begin{array}{rcl} K[B_{n,m}] & = & K[\{x\} \cup \{y_i, z_j, e_{i,j}, f_{i,j}\}_{1\leq i\leq n;\, 1\leq j\leq m}], \\ K[C_{n,m}] & = & K[\{x\} \cup \{a_i, b_j, e_{i,j}, f_{i,j}\}_{1\leq i\leq n;\, 1\leq j\leq m}], \\ K[D_{n,m}] & = & K[\{x\} \cup \{e_{i,j}, f_{i,j}\}_{1\leq i\leq n;\, 1\leq j\leq m}] \end{array}$$

over K. Let

$$\pi : K[\Psi_{n,m}] \to K[t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+m}, t^{-1}_{n+1}, \ldots, t^{-1}_{n+m}, s]$$

denote the homomorphism defined by setting

$$\pi(x) = s, \quad \pi(a_i) = t_i^2 s, \quad \pi(b_j) = t^2_{n+j} s, \quad \pi(y_i) = t_i s, \quad \pi(z_j) = t_{n+j} s,$$

$$\pi(e_{i,j}) = t_i t_{n+j} s, \quad \pi(f_{i,j}) = t_i t_{n+j}^{-1} s.$$

Thus $\pi(K[\Psi_{n,m}]) = \mathcal{R}_K[\Psi_{n,m}]$ and the kernel of π is the toric ideal $I_{\Psi_{n,m}}$.

Write $<_{lex}^{bbb}$ for the lexicographic order on $K[B_{n,m}]$ induced by the ordering of the variables

$$\begin{aligned} x &< y_1 < \cdots < y_n < z_1 < \cdots < z_m \\ &< e_{1,m} < f_{1,m} < e_{1,m-1} < f_{1,m-1} < \cdots < e_{1,1} < f_{1,1} \\ &< e_{2,m} < f_{2,m} < e_{2,m-1} < f_{2,m-1} < \cdots < e_{2,1} < f_{2,1} \\ &< \cdots \\ &< e_{n,m} < f_{n,m} < e_{n,m-1} < f_{n,m-1} < \cdots < e_{n,1} < f_{n,1}. \end{aligned}$$

Corollary 2.3. *The set of the binomials*

$$\begin{array}{rl} e_{i,\ell}e_{j,k} - e_{i,k}e_{j,\ell}, & i < j,\ k < \ell, \\ f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i < j,\ k < \ell, \\ e_{i,\ell}f_{j,k} - f_{i,k}e_{j,\ell}, & i < j,\ k < \ell, \\ f_{i,\ell}e_{j,k} - e_{i,k}f_{j,\ell}, & i < j,\ k < \ell, \\ y_i e_{j,k} - e_{i,k}y_j, & i < j, \\ e_{i,j}z_k - e_{i,k}z_j, & j < k, \\ y_i f_{j,k} - f_{i,k}y_j, & i < j, \\ e_{i,k}f_{j,k} - y_i y_j, & \\ f_{i,j}z_j - y_i x, & \\ x e_{i,j} - y_i z_j, & \end{array}$$

is a Gröbner basis of $I_{B_{n,m}}$ with respect to the lexicographic order $<_{lex}^{bbb}$.

Write $<_{lex}^{dd}$ for the lexicographic order on $K[D_{n,m}]$ which is obtained by restricting the lexicographic order $<_{lex}^{d}$ introduced in (1.4) to $\mathcal{M}(K[D_{n,m}])$.

Theorem 2.4. *The set of the binomials*

$$\begin{array}{rl} e_{i,\ell}e_{j,k} - e_{i,k}e_{j,\ell}, & i < j,\ k < \ell, \\ f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i < j,\ k < \ell, \\ f_{i,\ell}e_{j,k} - e_{i,k}f_{j,\ell}, & i < j,\ k < \ell, \\ f_{i,k}e_{j,\ell} - e_{i,\ell}f_{j,k}, & i < j,\ k < \ell, \\ e_{i,k}f_{j,k} - e_{i,m}f_{j,m}, & i \leq j,\ k < m, \\ f_{i,k}e_{j,k} - e_{i,m}f_{j,m}, & i < j,\ k \leq m, \end{array}$$

is a Gröbner basis of $I_{D_{n,m}}$ with respect to the lexicographic order $<_{lex}^{dd}$.

However, if $n \geq 2$, then the toric ideal $I_{C_{n,m}}$ cannot be generated by quadratic binomials. Thus, in particular, $I_{C_{n,m}}$ possesses no quadratic Gröbner basis if $n \geq 2$.

Proposition 2.5. *Let $n \geq 2$ and $m \geq 1$. Then, the toric ideal $I_{C_{n,m}}$ cannot be generated by quadratic binomials.*

Proof. The binomial $a_1 f_{2,1}^2 - f_{1,1}^2 a_2$ belongs to $I_{C_{n,m}}$. However, none of the quadratic monomials $a_1 f_{2,1}$, $f_{2,1}^2$, $f_{1,1}^2$ and $f_{1,1}a_2$ can appear in a binomial belonging to $I_{C_{n,m}}$. Hence the toric ideal $I_{C_{n,m}}$ cannot be generated by quadratic binomials. □

Remark 2.6. Let $n \geq 2$ and $m \geq 1$. Since the monomial

$$t_1 t_2 s = \frac{(t_1 t_{n+1}^{-1} s)(t_2^2 s)}{(t_2 t_{n+1}^{-1} s)} \notin \mathcal{R}_K[C_{n,m}]$$

belongs to the quotient field of $\mathcal{R}_K[C_{n,m}]$ and since $(t_1 t_2 s)^2 = (t_1^2 s)(t_2^2 s)$ belongs to $\mathcal{R}_K[C_{n,m}]$, $\mathcal{R}_K[C_{n,m}]$ is not normal. It is known that, in general, if a configuration $\mathcal{A}$ possesses a unimodular triangulation, then $\mathcal{R}_K[\mathcal{A}]$ is normal. Hence, it turns out that $C_{n,m}$ possesses no unimodular triangulation. Thus, in particular, $I_{C_{n,m}}$ has no squarefree initial ideal.

Write $<_{lex}^{cc}$ for the lexicographic order on $K[C_{n,m}]$ induced by the ordering of the variables

$$\begin{aligned} x &< f_{1,m} < \cdots < f_{1,2} < f_{1,1} < f_{2,m} < \cdots < f_{2,2} < f_{2,1} \\ &< \cdots < f_{n,m} < \cdots < f_{n,2} < f_{n,1} \\ &< e_{1,m} < \cdots < e_{1,2} < e_{1,1} < e_{2,m} < \cdots < e_{2,2} < e_{2,1} \\ &< \cdots < e_{n,m} < \cdots < e_{n,2} < e_{n,1} \\ &< b_m < \cdots < b_2 < b_1 < a_n < \cdots < a_2 < a_1. \end{aligned}$$

Theorem 2.7. *The set of the binomials*

$$\begin{array}{rl}
e_{i,\ell}e_{j,k} - e_{i,k}e_{j,\ell}, & i < j, \ k < \ell, \\
f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i < j, \ k < \ell, \\
f_{i,\ell}e_{j,k} - e_{i,k}f_{j,\ell}, & i < j, \ k < \ell, \\
f_{i,k}e_{j,\ell} - e_{i,\ell}f_{j,k}, & i < j, \ k < \ell, \\
e_{i,k}f_{j,k} - e_{i,m}f_{j,m}, & i \leq j, \ k < m, \\
f_{i,k}e_{j,k} - e_{i,m}f_{j,m}, & i < j, \ k \leq m, \\
a_i b_j - e_{i,j}^2, & \\
a_i x - e_{i,m}f_{i,m}, & \\
b_j f_{i,j} - x e_{i,j}, & \\
a_i e_{j,k}e_{j,\ell} - a_j e_{i,k}e_{i,\ell}, & i < j, \ k \leq \ell, \\
a_i f_{j,k}f_{j,\ell} - a_j f_{i,k}f_{i,\ell}, & i < j, \ k \leq \ell, \\
b_k e_{i,\ell}e_{j,\ell} - b_\ell e_{i,k}e_{j,k}, & i \leq j, \ k < \ell, \\
a_i e_{j,m}f_{j,m} - a_j e_{i,m}f_{i,m}, & i < j, \\
a_i e_{j,k}f_{j,\ell} - a_j e_{i,k}f_{i,\ell}, & i < j, \ k \neq \ell, \\
b_k e_{i,m}f_{j,m} - x e_{i,k}e_{j,k}, & i \leq j, \ k < m,
\end{array}$$

is a Gröbner basis of $I_{C_{n,m}}$ with respect to the lexicographic order $<_{lex}^{cc}$.

3. Computation of normalized volumes

We now turn to the problem of computing the normalized volume of each of the configurations $B_{n,m} \subset \mathbb{Z}^{n+m}$, $C_{n,m} \subset \mathbb{Z}^{n+m}$ and $D_{n,m} \subset \mathbb{Z}^{n+m}$. Then, the following lemma plays an important role.

A *noncrossing spanning subgraph* of the complete bipartite graph $\Sigma_{n,m}$ is a connected subgraph T of $\Sigma_{n,m}$ such that (i) the vertex set of T is $[n+m]$; (ii) if $\{i,k\}$ and $\{j,\ell\}$ with $i<j$ are edges of T, then $k \leq \ell$.

Lemma 3.1. *The number of noncrossing spanning subgraphs of the complete bipartite graph $\Sigma_{n,m}$ is*

$$\binom{n+m-2}{n-1}.$$

By virtue of Corollary 2.3, Theorem 2.4 and Theorem 2.7 together with Lemma 0.3 and Lemma 3.1, we have the following theorems. (We use the convention on binomial coefficients with $\binom{a}{0} = 1$ for all $a \in \mathbb{Z}$.)

Theorem 3.2. *Let $n \geq 1$ and $m \geq 1$, and let $\Sigma_{n,m}$ denote the complete bipartite graph on $[n+m]$ with the edges $\{i,j\}$, where $1 \leq i \leq n < j \leq n+m$. Let $B_{n,m}$ (resp. $D_{n,m}$) denote the configuration $B_{n+m}(\Sigma_{n,m})$ (resp. $D_{n+m}(\Sigma_{n,m})$).*

(a) *The normalized volume of $B_{n,m}$ is $\alpha+\beta$, where*

$$\alpha = \sum_{\substack{1\leq i\leq n \\ 1\leq k\leq \ell\leq m}} 2^{m-\ell}\binom{\ell-1}{k-1}\binom{i+m-k-1}{m-k},$$

$$\beta = \sum_{\substack{1\leq i\leq n \\ 1\leq k\leq m}} \binom{m}{k}\binom{i+k-2}{i-1}+1.$$

(b) *The normalized volume of $D_{n,m}$ is*

$$\sum_{\substack{1\leq i\leq n \\ 1\leq k\leq m}} \binom{m-1}{k-1}\binom{i+k-2}{i-1}\binom{n-i+m-k}{n-i}.$$

Theorem 3.3. *Let $n \geq 1$ and $m \geq 1$, and let $\Sigma_{n,m}$ denote the complete bipartite graph on $[n+m]$ with the edges $\{i,j\}$, where $1 \leq i \leq n < j \leq n+m$. Let $C_{n,m}$ denote the configuration $C_{n+m}(\Sigma_{n,m})$. Then, the normalized volume of $C_{n,m}$ is $\alpha+\beta+\gamma$, where*

$$\alpha = \sum_{\substack{1\leq i\leq n \\ 1\leq k\leq m}} \binom{m-1}{k-1}\binom{i+k-2}{i-1}\binom{n-i+m-k}{n-i},$$

$$\beta = \sum_{\substack{1\leq i\leq p\leq n \\ 1\leq k\leq m}} 2^{n-p}\binom{m-1}{k-1}\binom{i+k-2}{i-1}\binom{p-i+m-k}{p-i},$$

$$\gamma = \sum_{\substack{1\leq p\leq q\leq m \\ 1\leq i\leq n \\ 1\leq k\leq m-q+1}} 2^{p-1}\binom{q-1}{p-1}\binom{m-q}{k-1}\binom{i+k-2}{i-1}\binom{n-i+m-p-k}{n-i}.$$

Remark 3.4. The initial ideal $in_{<_{lex}^{cc}}(I_{C_{n,m}})$ is not quadratic. However, $\sqrt{in_{<_{lex}^{cc}}(I_{C_{n,m}})}$ is generated by quadratic monomials; in other words, the triangulation $\Delta(in_{<_{lex}^{cc}}(I_{C_{n,m}}))$ is a *flag complex*.

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